C3 Q102 LESSON 1

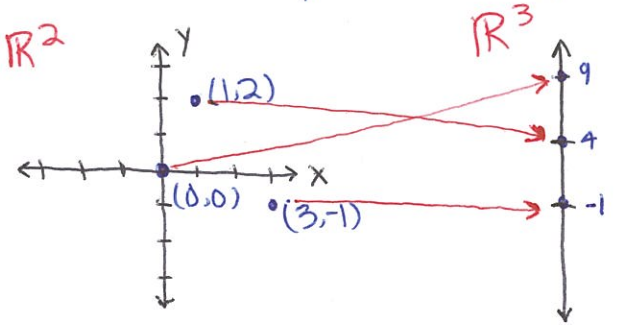
$\rightarrow f:\mathbb{R}^n \mapsto \mathbb{R}^1$

PART I: MULTIVARIABLE FUNCTIONS: $f:\mathbb{R}^2 \mapsto \mathbb{R}^1$

(INTRODUCTION AND ILLUSTRATIONS)

$\boxed{\text{Concept Development}}$: consider a function of type $z = f(x,y)$

ex. $f(x,y) = 9 - x^2 - y^2$

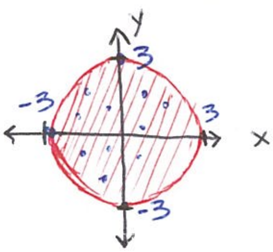


- An ordered pair from the cartesian plane yields a single one-dimensional value
- Since f takes any coordinate from $\mathbb{R}^2$, its natural domain is:

$$D:\{(x,y) \mid (x,y) \in \mathbb{R}^2\}$$

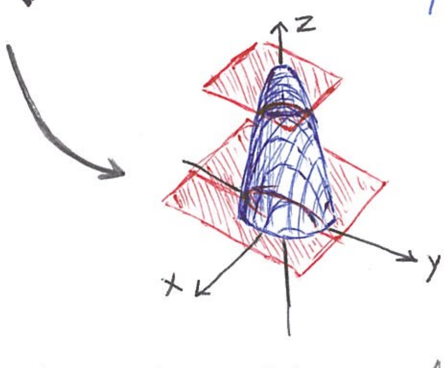
Now suppose the domain of the same function is restricted

$$f(x,y) = 9 - x^2 - y^2 \quad D:\{(x,y) \mid x^2 + y^2 \le 9\}$$

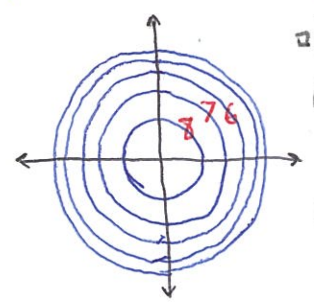


Take this restricted domain and let each point correspond to a z-value, such that it has a "height"

ex. $f(x,y)$ = height/altitude of some hill at point (x,y) of $\mathbb{R}^2$



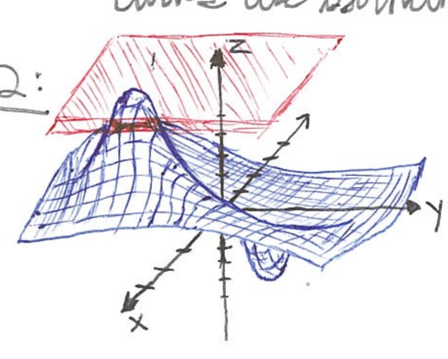
Take "slices" at different values of z $\longrightarrow$ and project the traces



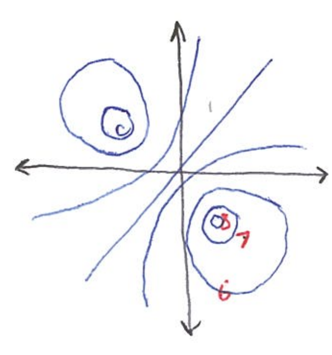
- Every point along a level curve remains constant

- Alternate interpretation: $f(x,y)$ = temp. of a plate at points (x,y), and the level curves are isothermals

Level curves Ex. 2:



$\longrightarrow$



PART II: FIRST PARTIAL DERIVATIVES

A. First Partial Derivatives at a Point

- Review | First derivatives in $\mathbb{R}^2$:
  - $f : \mathbb{R}^1 \mapsto \mathbb{R}^1$; function that maps from 1D to 1D

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \qquad h = \Delta x$$

- First Partial Derivative in $\mathbb{R}^3$:
  - $f : \mathbb{R}^2 \mapsto \mathbb{R}^1$ ex. $z = f(x,y)$

"pue z pue x" "derivative of f with respect to x"

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b) \qquad h = \Delta x$$

constant ← → changing incrementally

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b) \qquad h = \Delta y$$

1. Let $z = f(x, y)$. Assume f is continuous and differentiable.

The table below gives values of f at certain (x, y) coordinates.

y \ x	0	1	2	3	4	5
0	2.0	3.0	4.0	5.0	6.0	7.0
1	4.0	5.5	6.0	5.75	5.0	4.0
2	6.0	6.0	6.0	6.0	6.0	6.0
3	10.0	7.0	5.0	6.5	5.1	6.9
4	12.0	8.5	4.5	7.5	5.0	7.0
5	18.0	9.0	3.0	8.5	4.5	8.0

ossible nterpretations

Suppose $f(x, y)$ represents the temperature (°F) at points (x, y) where x and y are measured in inches.

OR

Suppose $f(x, y)$ represents the altitude of a range (miles) at points (x, y) where x and y are measured in km.

"What is the rate of change of f at (1,4) as we increase in x?"

Estimate and interpret: $f_x(1,4)$, $f_y(1,4)$, $f_x(4,2)$, $f_y(4,2)$.

ave rate Δ over nearest neighborhood

- $$f_x(1,4) = \lim_{\Delta x \to 0} \frac{f(1+\Delta x, 4) - f(1,4)}{\Delta x} \approx \frac{f(2,4) - f(0,4)}{2-0} = \frac{4.5-12}{2} = \boxed{-3.75}$$

Interpretation: the temp. (alt.) decreases approx. 3.75 °F/in. (mi/km) as we move in the $+x$ direction away from (1,4)

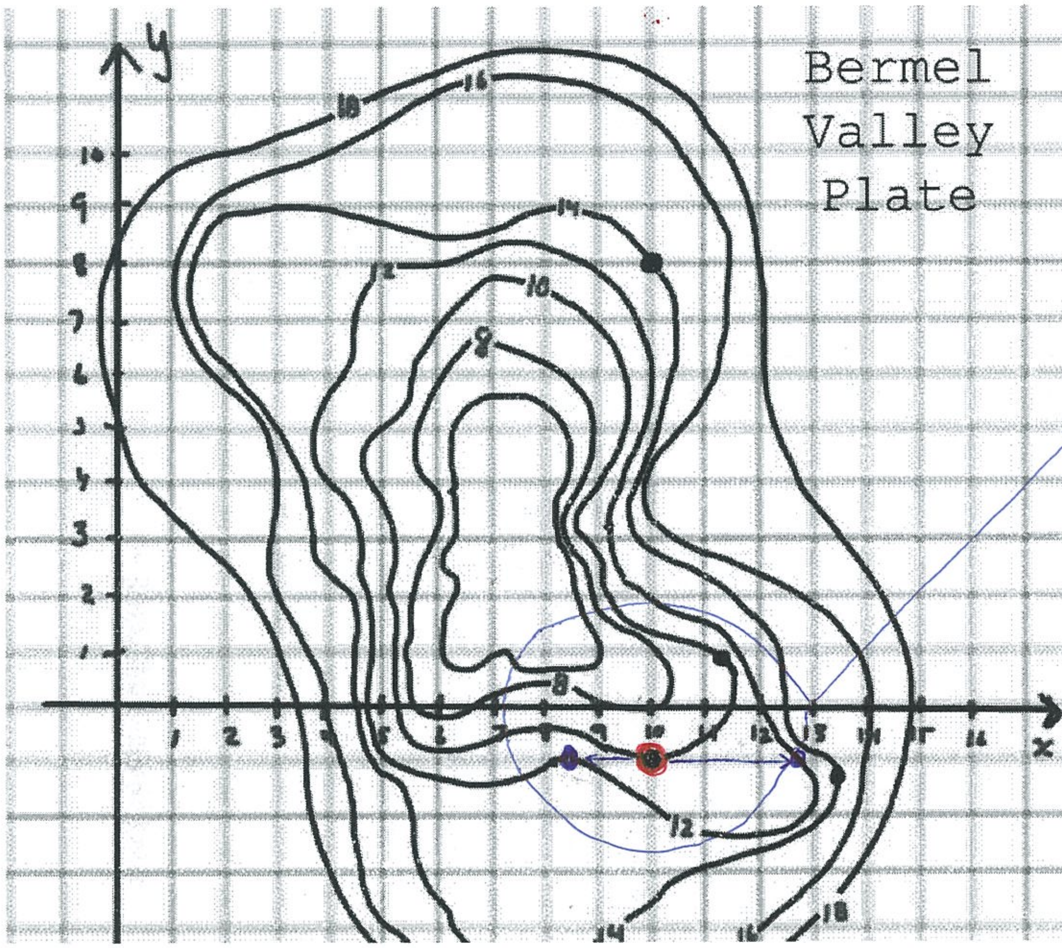
- $$f_y(1,4) = \lim_{\Delta y \to 0} \frac{f(1,4+\Delta y) - f(1,4)}{\Delta y} \approx \frac{f(1,5) - f(1,3)}{5-3} = \frac{9-7}{2} = \boxed{1}$$

- $$f_x(4,2) = \lim_{\Delta x \to 0} \frac{f(4+\Delta x, 2) - f(4,2)}{\Delta x} \approx \frac{f(5,2) - f(3,2)}{5-3} = \boxed{0}$$

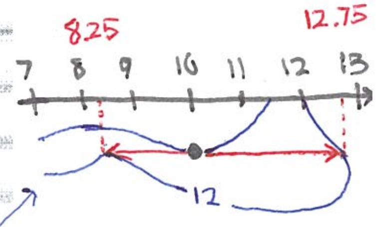
- $$f_y(4,2) = \lim_{\Delta y \to 0} \frac{f(4,2+\Delta y) - f(4,2)}{\Delta y} \approx \frac{f(4,3) - f(4,1)}{3-1} = \boxed{0.05}$$

2. Let $z = f(x,y)$. Assume f is continuous and differentiable.

The level curve map for f is given below.



graphical estimates



Suppose $f(x,y)$ represents the temperature (°F) at points (x, y) where x and y are measured in inches.

OR

Suppose $f(x,y)$ represents the altitude of a range (miles) at points (x, y) where x and y are measured in m.

"staying on the level curve as we move along the x-direction"

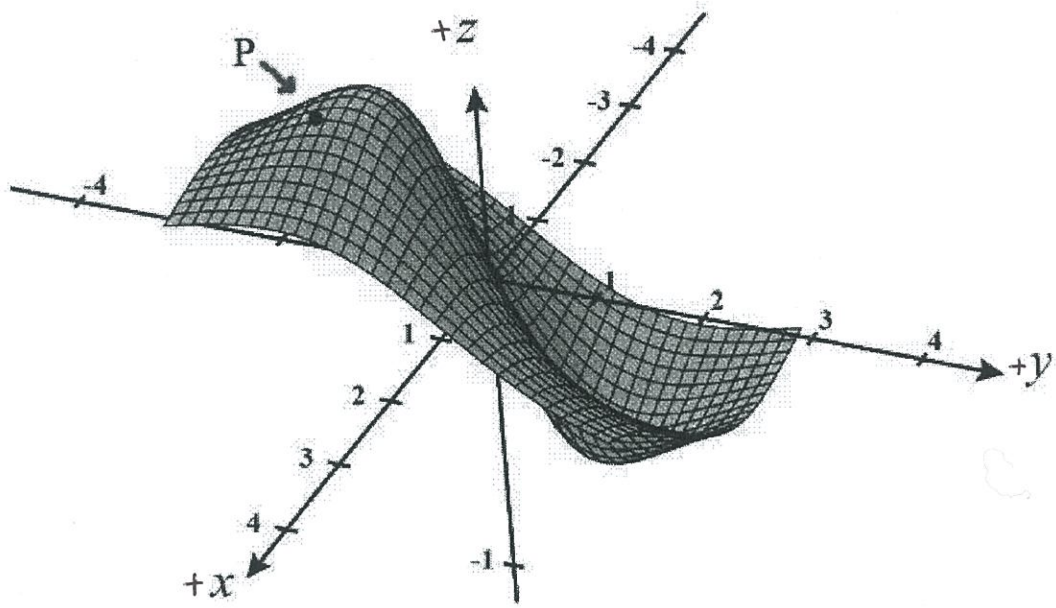
Estimate and interpret: $f_x(10,-1)$, $f_y(10,-1)$

- $f_x(10,-1) \approx \dfrac{f(12.75,-1)-f(8.25,-1)}{12.75-8.25} = \dfrac{12-12}{4.5} = 0$ °F/in or mi/m

- $f_y(10,-1) \approx \dfrac{f(10,0)-f(10,-1.75)}{0-(-1.75)} = \dfrac{8-12}{1.75} = -\dfrac{16}{7}$

3. Let $z = f(x, y)$. Assume f is continuous and differentiable.

The graph of f is shown below.

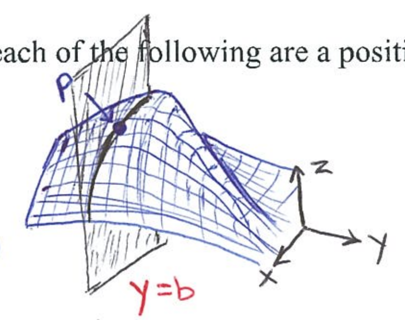


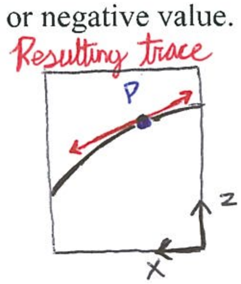
Suppose $f(x, y)$ represents the temperature (°F) at points (x, y) where x and y are measured in inches.

OR

Suppose $f(x, y)$ represents the altitude of a range (miles) at points (x, y) where x and y are measured in m.

Determine if each of the following are a positive or negative value.

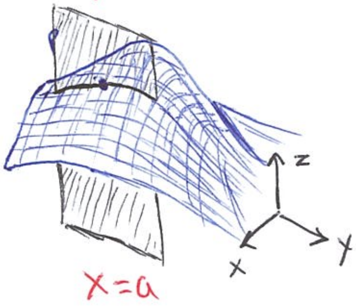


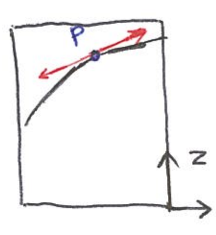


f_x at point P.

$f_x|_P < 0$

- As x increases when y is held constant at $y=b$, the function goes down; the slope of the tangent is negative





f_y at point P.

$f_y|_P > 0$

- As y increases at $x=a$, the function goes up and the tangent is positive

B. First Partial Derivative Functions

Definition: Let f be a function of x and y.

The **first partial derivative of f with respect to x** is the function

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$ provided it exists.

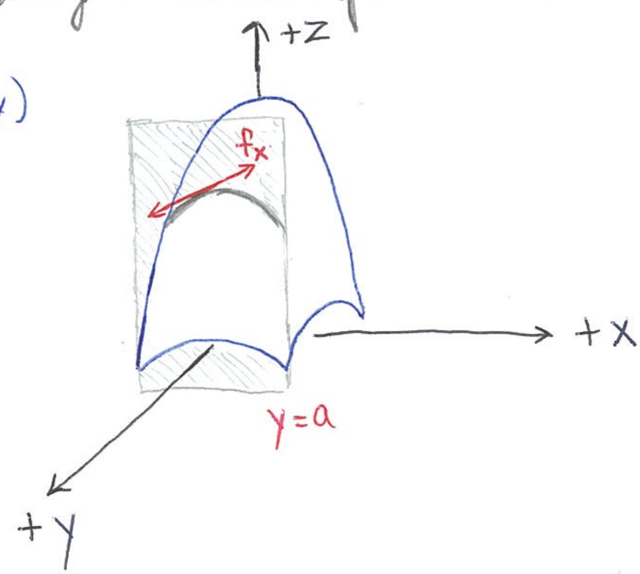
The **first partial derivative of f with respect to y** is the function

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$ provided it exists.

Notation: $f_x(x,y) = f_x = \frac{\partial f(x,y)}{\partial x} = \frac{\partial f}{\partial x} = D_x f$

- This function represents the rate of change in f as x increases if we hold y constant

i.e. Along any plane $y=b$, the slope of the tangent is this function f_x



First Partial Derivative Techniques

Example1: $f(x, y) = x^3y^2 - 2x^2y + 3x$

Find the first partial derivatives.

★Since y is held constant, the y terms are treated as coefficients to the x-terms

$$f_x = \frac{\partial f}{\partial x} = y^2 \cdot \frac{d}{dx}(x^3) - y \cdot \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x)$$

$$= y^2(3x^2) - y(4x) + 3 = \boxed{3x^2y^2 - 4xy + 3}$$

$$f_y = \frac{\partial f}{\partial y} = \boxed{2x^3y - 2x^2}$$

Example2: $z = xy^2e^{xy}$

Find the first partial derivatives.

$$\frac{\partial z}{\partial x} = y^2(x \cdot e^{xy} \cdot y + e^{xy} \cdot y^2)$$

$$\frac{\partial z}{\partial y} = x(y^2 \cdot e^{xy} \cdot x + e^{xy} \cdot x \cdot 2y)$$

C3 Q102 LESSON 2

PART I: GRADIENT AND DIRECTIONAL DERIVATIVE $f:\mathbb{R}^2 \mapsto \mathbb{R}^1$

GRADIENT $f:\mathbb{R}^2 \mapsto \mathbb{R}^1$

Del Definition

$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ → Vector operator (acts on something else)

Result of Del Operating on f: the Gradient Vector

$\nabla f = \nabla f(x,y) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle f(x,y)$ → scalar function

Gradient: $= \boxed{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}$

- Vector of corresponding x and y slopes
- Direction of greatest change in f
- "Natural slope" vector

Magnitude of Gradient

$|\nabla f| = \sqrt{f_x^2 + f_y^2}$ → max rate of change in f

DIRECTIONAL DERIVATIVE $f:\mathbb{R}^2 \mapsto \mathbb{R}^1$

- previously...

f_x = rate of change in f when moving in the $+x$ direction $\rightarrow \langle 1,0 \rangle = \hat{i}$

f_y = rate of change in f when moving in the $+y$ direction $\rightarrow \langle 0,1 \rangle = \hat{j}$

- now...

The Directional Derivative is the rate of change in f when moving in direction $\vec{v}$, such that $\vec{v} \neq \hat{i}, \hat{j}$

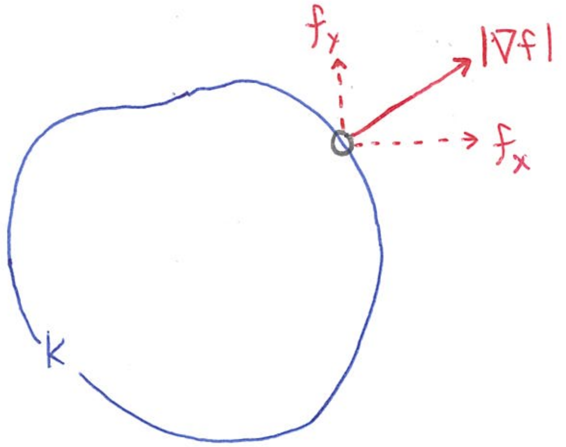
↳ this is equal to the component of ∇f along $\vec{v}$

Therefore, the directional derivative of f in the direction of $\vec{v}$

$$= \boxed{\mathrm{Comp}_{\vec{v}} \nabla f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|} = \nabla f \hat{u} = D_{\hat{u}} f}$$

(*more on directional derivative*)

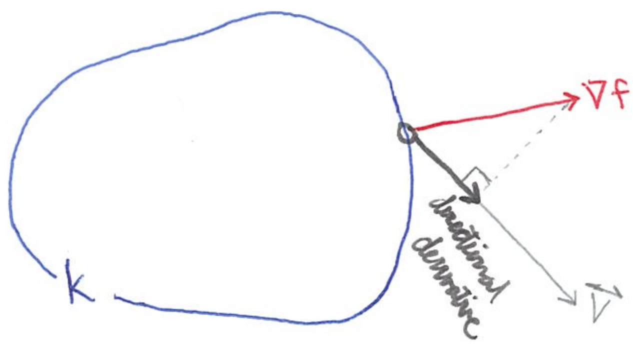
Visual: Consider the level curve $f(x,y)=K$



f_x = instanous rate of change w.r.t. x

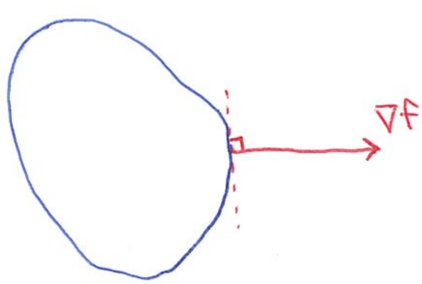
f_y = instaneous rate of change w.r.t. y

$|\nabla f|$ = max instaneous rate of change in f



- The directional derivative is the component of ∇f onto $\vec{v}$ (how much of ∇f is applied onto vector $\vec{v}$)

Property: ∇f at P is always orthogonal to the level curve at P



Intuition:

- if we were to move in the direction of the tangent, then we are essentially moving along the level curve
  - ↳ it then follows that there would be no change in f
- We can reasonably infer that the max rate of change would occur when we move perpendicular to the tangent

Example 1: Let $f(x,y)=x^3y^2$

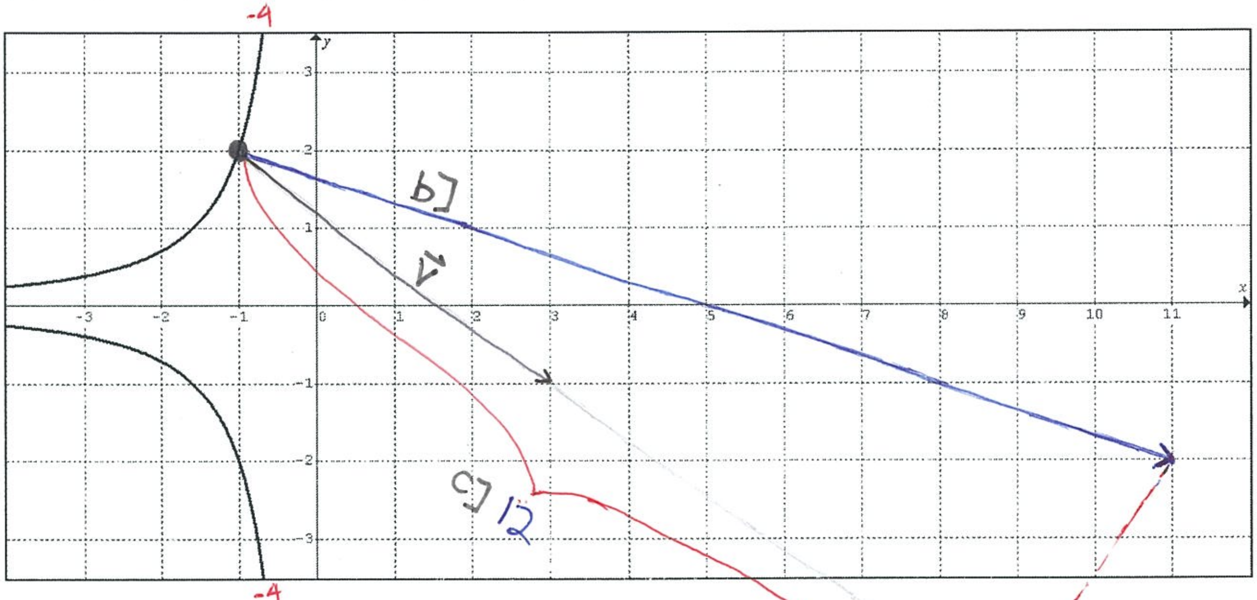
(a) Define the level curve shown below which passes through the point (-1, 2).

(b) In what direction from the point (-1, 2) will f have a maximum rate of change?

(c) What is the rate of change in f from (-1, 2) in the direction of $\mathbf{v}=\langle 4,-3\rangle$?

(d) If $f(x,y)$ is temperature ($^\circ F$) at (x,y) and x and y are in meters, provided an interpretation of the answer found in part (c).

(e) Graphically illustrate parts (b) and (c).



a] $f(-1,2)=(-1)^3(2)^2=-4$ → the graph is defined by the level curve $x^3y^2=-4$

b] $\nabla f=\left\langle \frac{\partial}{\partial x}[x^3y^2], \frac{\partial}{\partial y}[x^3y^2]\right\rangle=\langle 3x^2y^2, 2x^3y\rangle$

$\nabla f\big|_{(-1,2)}=\langle 3(-1)^2(2)^2, 2(-1)^3(2)\rangle=\boxed{\langle 12,-4\rangle}$

c] $\text{Comp}_{\vec{v}}\nabla f=\dfrac{\nabla f\cdot\vec{v}}{\|\vec{v}\|}=\dfrac{\langle 12,-4\rangle\cdot\langle 4,-3\rangle}{\sqrt{4^2+3^2}}=\dfrac{48+12}{5}=12$

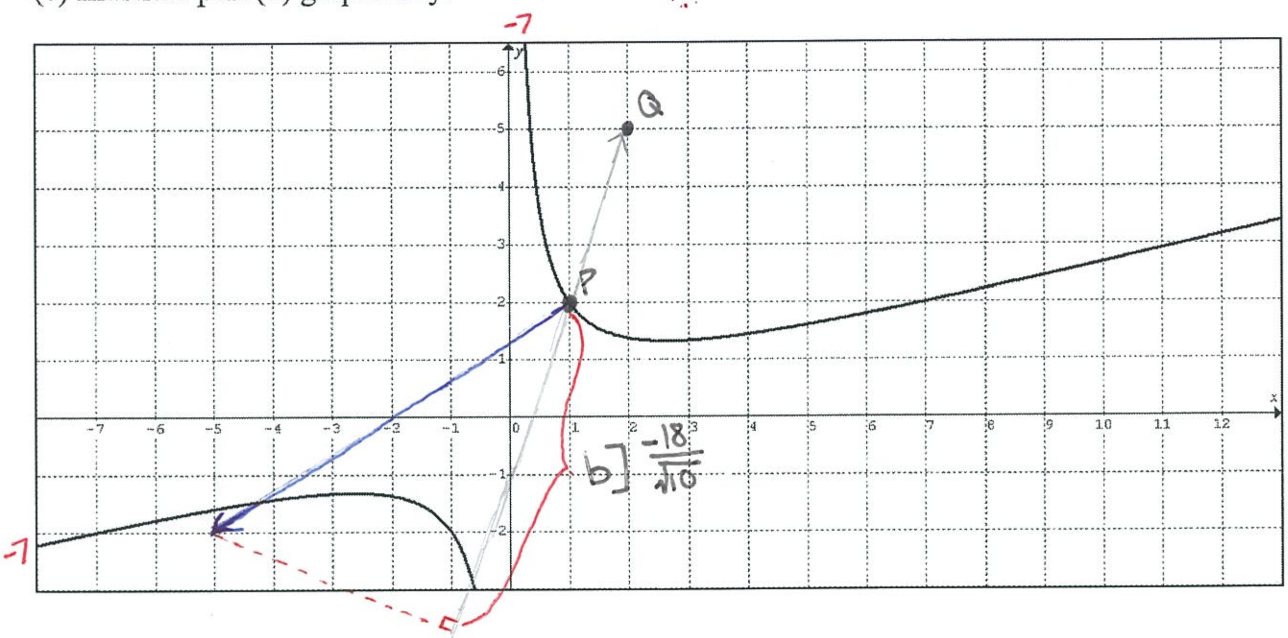
d] The temperature will increase at a rate of 12°F/m as we move in the direction ⟨4,-3⟩ from the point (-1,2)

Example 2: Let $f(x,y)=x^2-4xy$

(a) Define the level curve shown below which passes through the point (1, 2)

(b) Find the directional derivative of f at P(1,2) in the direction from P(1,2) to Q(2,5).

(c) Illustrate part (b) graphically.



a] $f(1,2)=(1)^2-4(1)(2)=-7 \rightarrow f(x,y)=x^2-4xy=-7$

b] $\nabla f=\langle 2x-4y,-4x\rangle$

$\nabla f|_{(1,2)}=\langle -6,-4\rangle \qquad \overrightarrow{PQ}=\langle 2-1,5-2\rangle=\langle 1,3\rangle=\vec{v}$

$D_{\hat{u}}f(1,2)=\dfrac{\langle -6,-4\rangle\cdot\langle 1,3\rangle}{\sqrt{1^2+3^2}}=\dfrac{-6-12}{\sqrt{10}}=\dfrac{-18}{\sqrt{10}}$

Example 3: Let $f(x,y) = 2 + x^2 + \frac{1}{4}y^2$

(a) Define the level curve of f that passes through the point P(1,2)

(b) Find the direction in which $f(x,y)$ increases most rapidly at the point P(1,2).
Find the direction in which $f(x,y)$ decreases most rapidly at the point P(1,2).

(c) Find the maximum rate of change of f at point P.
Find the minimum rate of change of f at point P.

a] $f(1,2) = 2 + 1^2 + \frac{1}{4}(2)^2 = 4$

Ellipse: center $(0,0)$

Level curve: $2 + x^2 + \frac{1}{4}y^2 = 4 \Rightarrow \frac{x^2}{2} + \frac{y^2}{\sqrt{8}} = 1$ $\quad r_x = \sqrt{2}$, $r_y = 8^{1/4}$

b] $\nabla f = \langle 2x, \frac{1}{2}y \rangle$

$\nabla f|_{(1,2)} = \langle 2,1 \rangle \rightarrow$ direction for max rate of change

$-\nabla f|_{(1,2)} = -\langle 2,1 \rangle = \langle -2,-1 \rangle \rightarrow$ direction for min rate of change

c] $|\nabla f|_{(1,2)}| = \sqrt{5} \rightarrow$ max rate of change

$-|\nabla f|_{(1,2)}| = -\sqrt{5} \rightarrow$ min rate of change

PART II: THE CHAIN RULE

Consider the function $z = (x^2+y^2)^3 + (5xy)^2$

- The Chain Rule is required to differentiate this <u>composite</u> function

method ① <u>"Direct" Differentiation - for simpler functions</u>

product rule and chain rule

a] $\frac{\partial z}{\partial x} = 3(x^2+y^2)^2(2x) + 2(5xy)\cdot 5y$

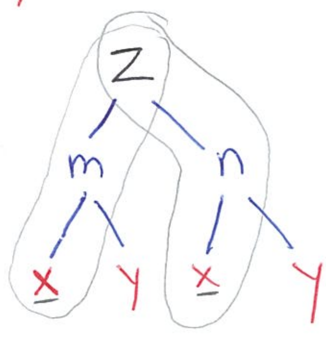
b] $\frac{\partial z}{\partial y} = 3(x^2+y^2)^2(2y) + 2(5xy)\cdot 5x$

method ② <u>The Chain Rule for Partial Derivatives</u>

- to redefine this problem to use the chain rule, we can introduce arbitrary variables

Let $z = m^2 n \qquad m = x^2+y^2 \qquad n = 5xy$

a] $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial z}{\partial n}\cdot\frac{\partial n}{\partial y}$



have to "pass through" m/n before you can get to x/y

$= 2mn\cdot 2x + m^2 \cdot 5y$

$= 2(x^2+y^2)\cdot(5xy)\cdot 2x + (x^2+y^2)^2(5y)$

- Revisiting ①a]

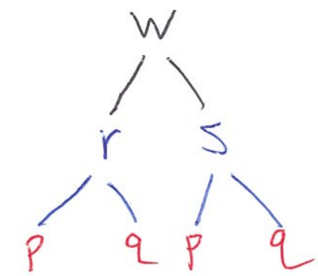
$$\frac{\partial z}{\partial x} = \underbrace{3(x^2+y^2)}_{\frac{\partial z}{\partial m}} \cdot \underbrace{(2x)}_{\frac{\partial m}{\partial x}} + \underbrace{2(5xy)}_{\frac{\partial z}{\partial n}} \cdot \underbrace{5y}_{\frac{\partial n}{\partial y}}$$

(chain rule continued)

ex.1 Let $w = F(r,s)$ $r = f(p,q)$ $s = g(p,q)$

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial p} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial p}$$

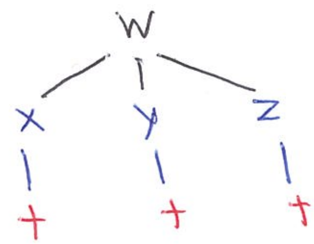
$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial q} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial q}$$



▫ Bottom "Rung" gives variables of interest

ex.2 Let $w = f(x,y,z)$ $x = g(t)$ $y = f(t)$ $z = h(t)$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$



$\downarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

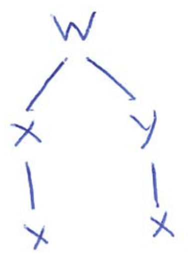
☆ $\frac{\partial w}{\partial t} = \frac{dw}{dt}$ is only true if we did not pass through x, y, z

IMPLICIT DIFFERENTIATION AND PARTIAL DERIVATIVES

Consider $\underbrace{x^2 y + 5xy^3 - x}_{\text{can be thought of as a function of two variables}} = 8$, find $\dfrac{dy}{dx}$:

$\rightarrow f(x,y) = 8 \Rightarrow \underbrace{f(x,y) - 8}_{F(x,y)} = 0$

$F(x,y) = 0$



Let $w = F(x,y) = 0 \quad x = x \quad y = f(x)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} \cdot \cancel{\frac{dx}{dx}}^{\,1} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}}$$

- $F_x = 2xy + 5y^3 - 1$
- $F_y = x^2 + 15xy^2$

$\longrightarrow \dfrac{dy}{dx} = -\dfrac{2xy + 5y^3}{x^2 + 15xy^2}$

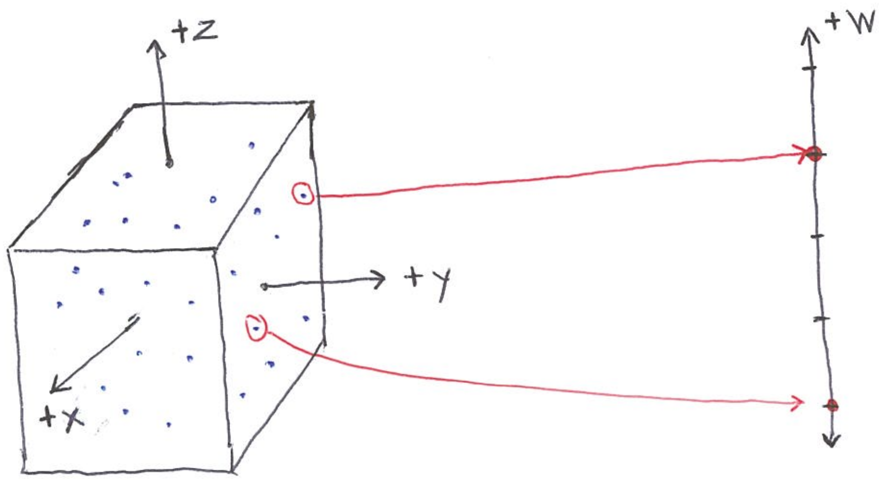
C3 Q102 LESSON 3

FUNCTIONS IN R4 (GENERAL OVERVIEW)

$w = f(x, y, z)$ $f : \mathbb{R}^3 \mapsto \mathbb{R}^1$

$(x, y, z) \rightarrow (w)$

- Functions in $\mathbb{R}^4$ cannot be visualized; no graph
- However, they can be contextualized by scenarios where every point in 3D space is assigned a value



☆ Each input, a point in the box, outputs a value onto the w scale

Possible Interpretations

w = temperature at point (x, y, z)

w = density at point (x, y, z)

LEVEL SURFACE (GENERAL NOTES)

Functions in $\mathbb{R}^3$ vs. $\mathbb{R}^4$

- $\mathbb{R}^3$: Function that maps from $\mathbb{R}^2 \mapsto \mathbb{R}^1$
  - graph in $\mathbb{R}^3$ — ex. $z=f(x,y)$
  - level curve in $\mathbb{R}^2$

- $\mathbb{R}^4$: Function that maps from $\mathbb{R}^3 \mapsto \mathbb{R}^1$
  - "graph" in $\mathbb{R}^4$ — ex. $w=f(x,y,z)$
  - level surface in $\mathbb{R}^3$

☆ Level surface of $w=f(x,y,z)$ is the equation:

- $f(x,y,z)=k$ for some constant k

↳ "what the function looks like when the output is k"

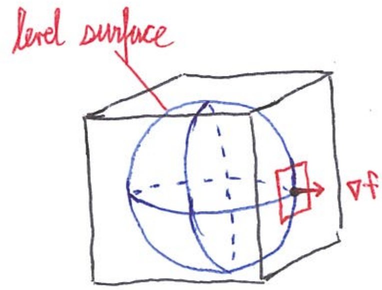
Gradients in $\mathbb{R}^3$ vs. $\mathbb{R}^4$

- $\mathbb{R}^3: f:\mathbb{R}^2 \mapsto \mathbb{R}^1$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

- $\mathbb{R}^4: f:\mathbb{R}^3 \mapsto \mathbb{R}^1$

☆ $$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$



∇f points in the direction of greatest increase

$|\nabla f|$ is the max rate of increase

GRADIENT AND DIRECTIONAL DERIVATIVE

EXAMPLE: $w = x^2 - y^2 + z^2$: $w = f(x, y, z)$

(1) Describe the Shape of the Graph of $w = x^2 - y^2 + z^2$

no shape $(\mathbb{R}^4)$

(2) Define the Domain of $w = x^2 - y^2 + z^2$

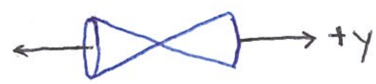
no restrictions ; $D: \{(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\}$

Let w = the temperature (°F) at point (x, y, z).

(3) Define the level surfaces at $w = 0, 4, -4$.

- $W = 0$ $\quad 0 = x^2 - y^2 + z^2$

  cone opening in y



- $W = 4$ $\quad 4 = x^2 - y^2 + z^2$

  hyperboloid of one sheet opening in y

- $W = -4$ $\quad 4 = -x^2 + y^2 - z^2$

  hyperboloid of two sheets opening in y

$w = f(x, y, z) = x^2 - y^2 + z^2$

(4) Compute the Gradient $\nabla f(x, y, z)$

$$\nabla f = \langle 2x, -2y, 2z \rangle$$

(5) Define the level surface that passes through the point (-1, 1, -2).

$$f(-1,1,2) = (-1)^2 - (1)^2 + (-2)^2 = 4$$

$$4 = x^2 - y^2 + z^2$$

hyperboloid of one sheet

(6) In what Direction from point (-1, 1, -2) will w increase must rapidly? What is the maximum rate of increase leaving point (-1, 1, -2)?

$$\nabla f(-1,1,-2) = \langle -2, -2, 4 \rangle$$

direction of greatest increase in temperature

$$|\nabla f(-1,1,-2)| = \sqrt{(-2)^2 + (-2)^2 + 4^2} = 2\sqrt{6}\ ^\circ F/m$$

(7) Find the Directional Derivative when moving from point P(-1, 1, -2) to point Q(4,2,-3).

$$\overrightarrow{PQ} = \langle 5, 1, -1 \rangle = \vec{v}$$

$$\text{Comp}_{\vec{u}} \nabla f = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle -2, -2, -4 \rangle \cdot \langle 5, 1, -1 \rangle}{\sqrt{5^2 + 1^2 + 1^2}} = \frac{-10 - 2 + 4}{\sqrt{27}}$$

$$= \frac{-8}{3\sqrt{3}}$$

TANGENT PLANE

elliptical cone

EXAMPLE: Find an equation of the plane tangent to the surface $x^2 + 3y^2 - z^2 = 0$ at the point (1, 1, 2) on the surface.

- Treat the equation as one level curve of a greater function w
  - ↳ this induces a four-dimensional function

$$w = x^2 + 3y^2 - z^2$$

↳ now we can set $w = 0$, such that the cone becomes one of many level surfaces

$$x^2 + 3y^2 - z^2 = 0$$

- finding the gradient of this equation allows us to find the direction of a vector orthogonal to the surface, which is needed to define the plane

$$\nabla w = \langle 2x, 6y, -2z \rangle$$

$$\nabla w|_{(1,1,2)} = \langle 2, 6, -4 \rangle$$ → normal vector to the surface at (1,1,2).

$$\boxed{\begin{gathered} 2(x-1) + 6(y-1) - 4(z-2) = 0 \\ x + 3y - 2z = 0 \end{gathered}}$$

C3 Q102 LESSON 4

PART I: SECOND PARTIAL DERIVATIVES

A. Notation and Clairaut's Theorem

Clairaut's Theorem: Let f be a function of two variables x and y. If f, f_x, f_y, f_{xy}, f_{yx} are continuous on an open interval region R, then $f_{xy} = f_{yx}$ throughout R.

2nd Partial Derivatives

- $\dfrac{\partial(f_x)}{\partial x} = \dfrac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial x} = (f_x)_x = f_{xx}$

- $\dfrac{\partial(f_x)}{\partial y} = \dfrac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$

- $\dfrac{\partial(f_y)}{\partial x} = \dfrac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$

always equal

B. MEANING OF SECOND PARTIALS

Let $z = f(x,y)$

$f_{xx} = \frac{\partial}{\partial x} f_x(x, y=b)$ → as x increases, how does f_x change?

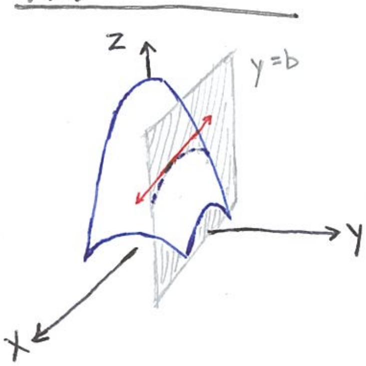
changing

$f_{yy} = \frac{\partial}{\partial y} f_y(x=a, y)$ → as y increases, how does f_y change?

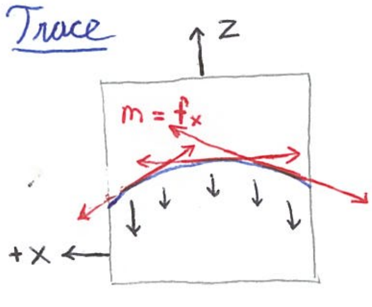
$f_{xy} = \frac{\partial}{\partial y} f_x(x, y=b)$ → as y increases, how does f_x change?

$f_{yx} = \frac{\partial}{\partial x} f_y(x=a, y)$ → as x increases, how does f_y change?

f_{xx} Visual

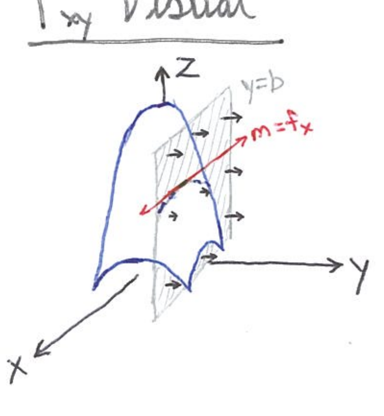


Trace



- the slope of each tangent line is represented by f_x at a given value of x
- in this case, f_x becomes more negative as we move in the $+x$ direction
  ∴ $f_{xx} < 0$; note that this checks out with the concavity of the trace

f_{xy} Visual



- f_{xy} is how f_x changes as b increases (as the tangent plane shifts forward)

EX: $f(x,y) = x^3y^2 - 2x^2y + 3x$ Find all second partial derivatives.

$$\underline{f_x} = 3x^2y^2 - 4xy + 3 \qquad \underline{f_y} = 2x^3y - 2x^2$$

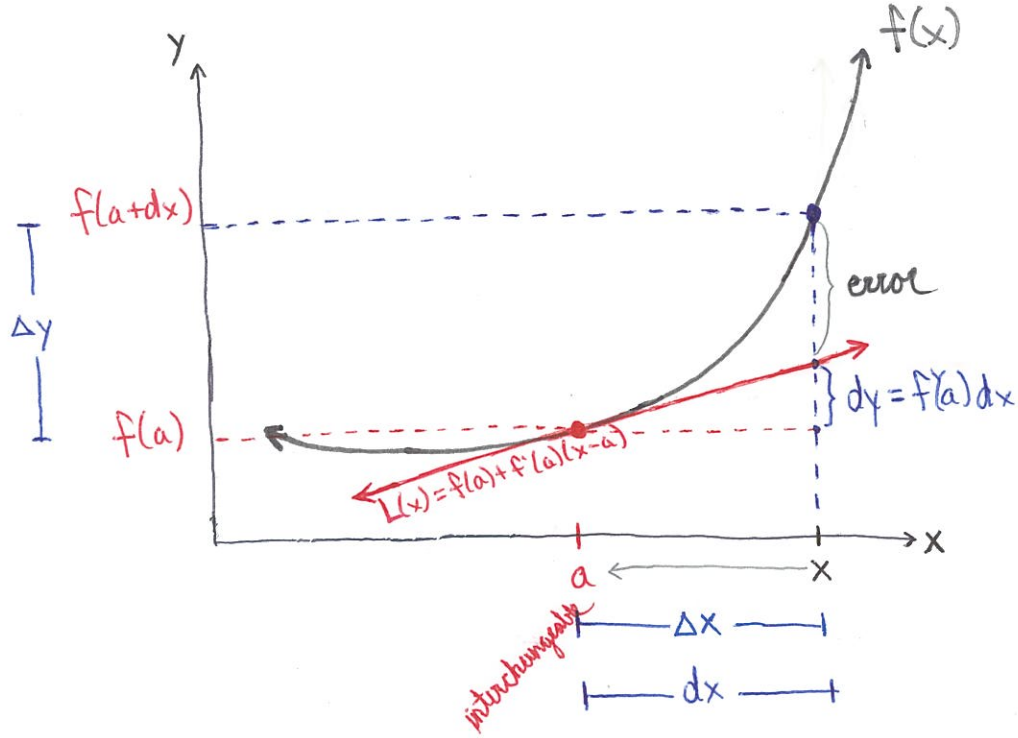
$$f_{xx} = \frac{\partial(f_x)}{\partial x} = \frac{\partial}{\partial x}\left[3x^2y^2 - 4xy + 3\right] = 6xy^2 - 4y$$

$$f_{yy} = 2x^3$$

$$\left.\begin{aligned} f_{xy} &= 6x^2 - 4x \\ f_{yx} &= 6x^2 - 4x \end{aligned}\right\} \text{equal by Clairaut's Thm}$$

PART II: TANGENT PLANE LINEARIZATION

Review: Tangent Line Linearization



$$\Delta y \approx dy = f'(a)dx$$

$$= f'(x)dx$$

generalized

$\cancel{\Delta y = f'(x)\, dx}$ wrong

$$\Delta y = f'(x)dx + \text{error}$$

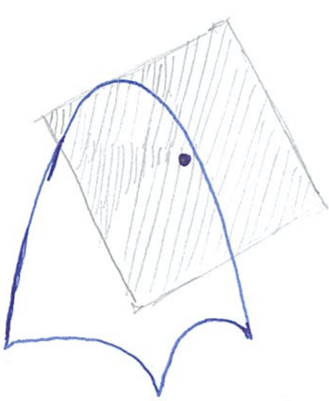
as $\Delta x \to 0$ ...

$$\cancel{\Delta}^{dy} y = f'(x)dx + \cancel{\text{error}}^{0}$$

$$\boxed{dy = f'(x)\, dx}$$

Tangent Plane Linearization

$z = f(x, y)$



$$\Delta f = f_x \Delta x + \text{error}_1 + f_y \Delta y + \text{error}_2$$

as $\Delta x, \Delta y \to 0$ ...

$\Delta x \to dx \quad \Delta y \to dy \quad \text{error} \to 0$

$$\boxed{df = f_x dx + f_y dy}$$

similar to Chain Rule

$$\frac{dz}{dx} = f_x \frac{dx}{dx} + f_y \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$